Failure in geotechnical Finite Element models: Second Order Work Criterion for Beam Elements

Gruner Group

ESGI'120

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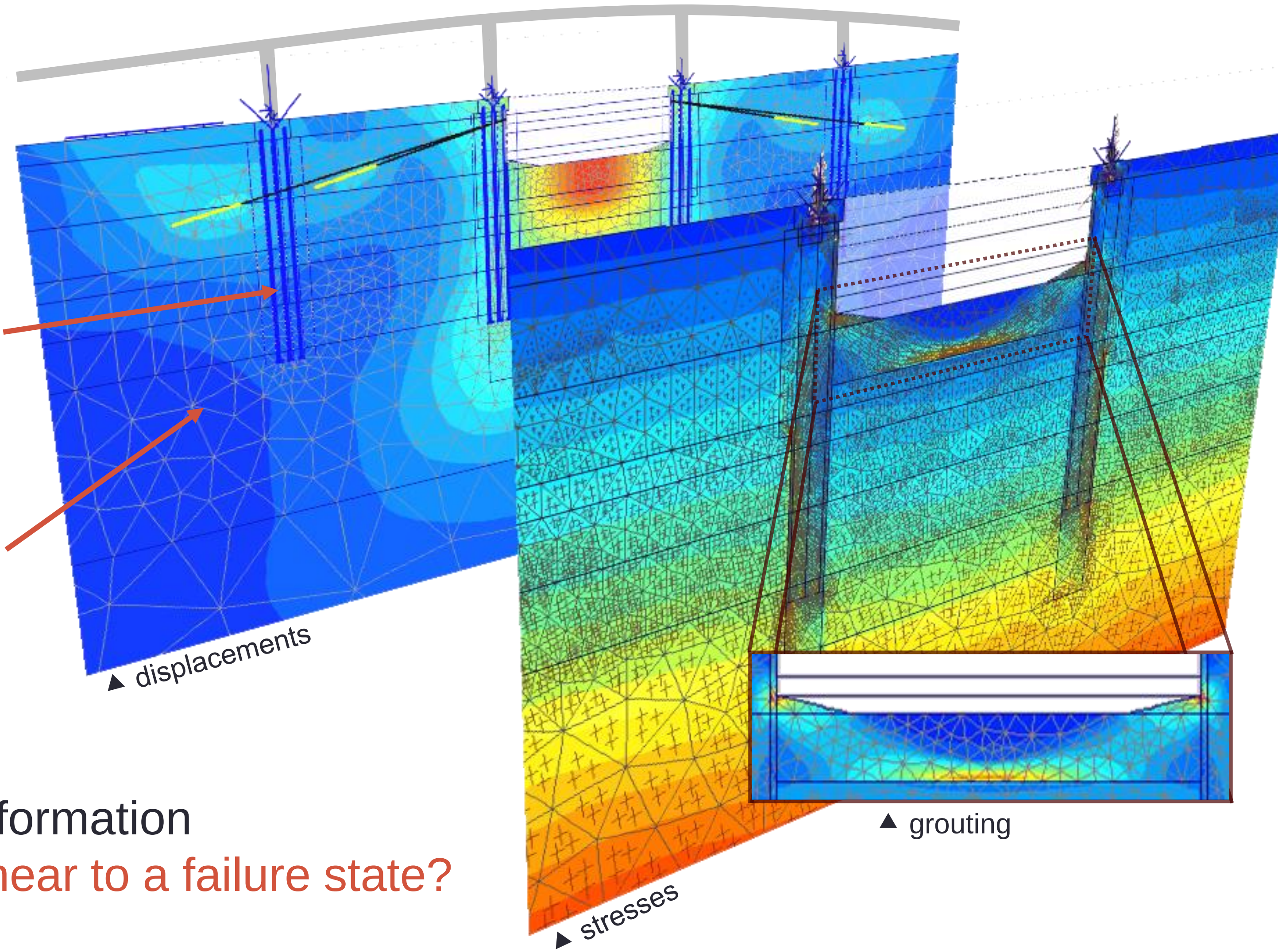
Geotechnical Finite Element Simulations

Excavation pit with grouting in the area of a bridge

“2D” Plain-Strain Model

Beam Elements (based on Mindlin’s theory of plates.)

Volume Elements

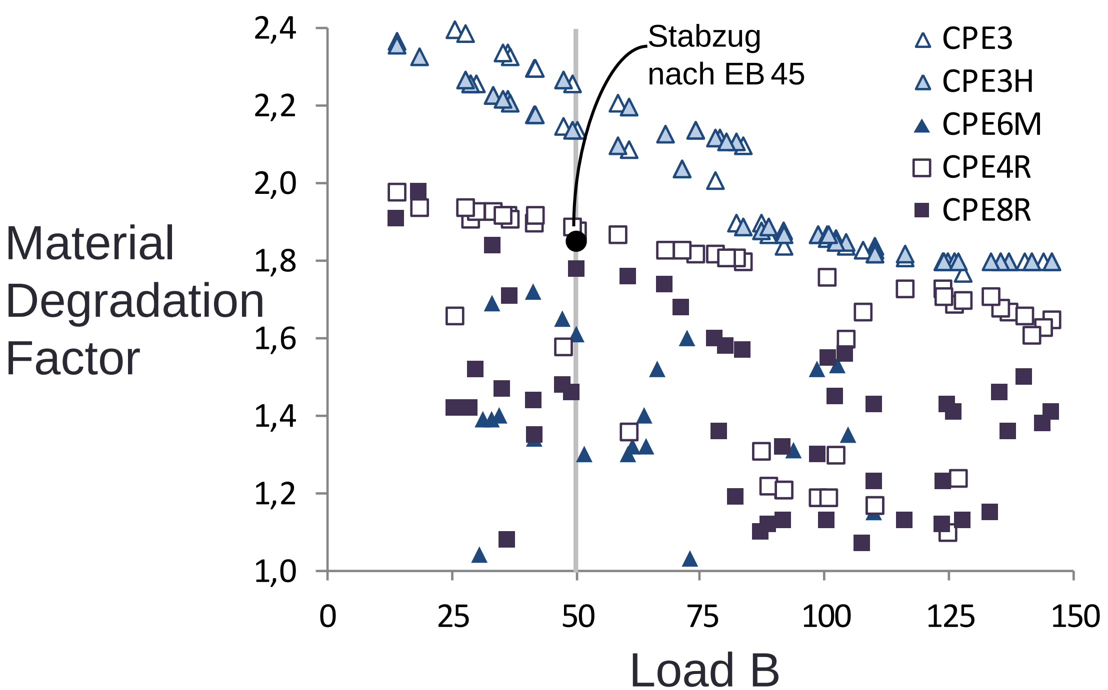


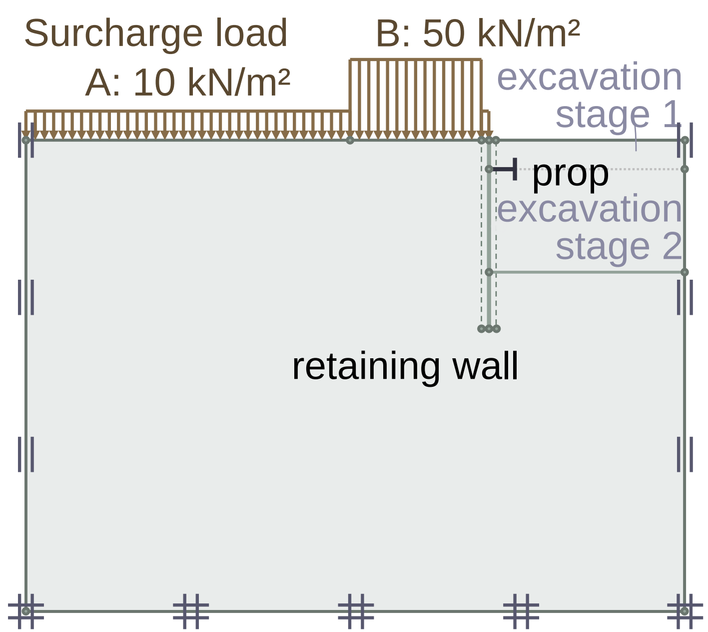
Objectives:

1. Prediction of Deformation
2. Is the structure near to a failure state?

How to define Failure for FE Simulations?

- Lack of numerical convergence? → Not good! (see below)
- Simulated deformation exceeds a given threshold? → 😐
- Hill‘s Criterion based on Second Order Work?





Hill‘s Criterion: Second Order Work (1)

Well-known criterion for failure (bifurcation in the solution) for models using elasto-plastic formulation: Hill’s failure criterion based on the second order work

Second order work at one material point:

$$u = d^2W = d\sigma : d\varepsilon$$

Hill’s condition of stability (Hill 1958):

$$d^2W > 0$$

Second Order Work for Plaxis-Beams?

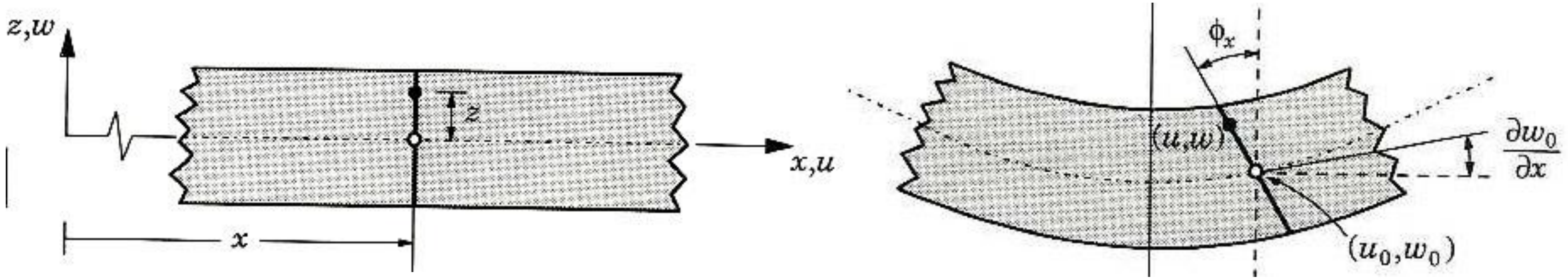
- Plaxis: Second Order Work **not given**
- Worse, only a small set of output variables at the beam element nodes are provided:
  - u_x: total nodal displacements in direction x
  - u_y: total nodal displacements in direction y
  - ϕ: total nodal rotation
  - N: Normal force extrapolated to the node
  - G: Shear force extrapolated to the node
  - M: Bending moment extrapolated to the node
- Objective: Calculate the second order work for this kind of beam elements as a post-processing after finishing the FE-simulation based only on the nodal variables provided in the programs output

Expected Results

- Show a way to calculate the **strain energy** of one beam element

- Show a way to calculate the **second order work** for this kind of beam element with the data given

Timoshenko beam model



Displacement fields: longitudinal *u*(*x*,*z*,*t*) and transverse *w*(*x*,*z*,*t*), based on Timoshenko's beam theory are expressed by the displacements on the middle line and the rotation of the cross section:

$$u(x,z,t) = u_0(x,t) + z\,\phi_y(x,t)$$
$$w(x,z,t) = w_0(x,t)$$

u_0, w_0 are the displacements on the middle line ($y = z = 0$).

ϕ_y is the rotation of the cross section

Timoshenko beam model

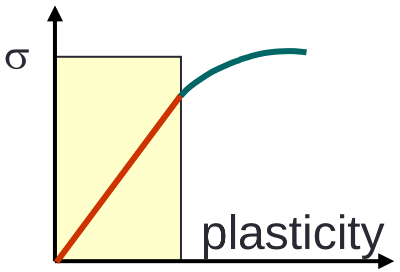
- Linear strain-displacement relations are assumed:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_y}{\partial x}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w_0}{\partial x} + \phi_y$$

ε_x is the direct strain

γ_{xz} is the shear strain



- Hooke's law is used to express the linear stress-strain relations

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 \\ 0 & \lambda G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} = \boldsymbol{D}\,\boldsymbol{\varepsilon}$$

σ_x is direct stress in x direction,
τ_{xz} is shear stress
E is Young modulus,
G is shear modulus,
λ is shear correction factor.

Timoshenko beam model

The strain energy has the following form:

$$U = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}dV = \frac{1}{2}\int_V (\varepsilon_x\sigma_x + \gamma_{xz}\tau_{xz})dV$$

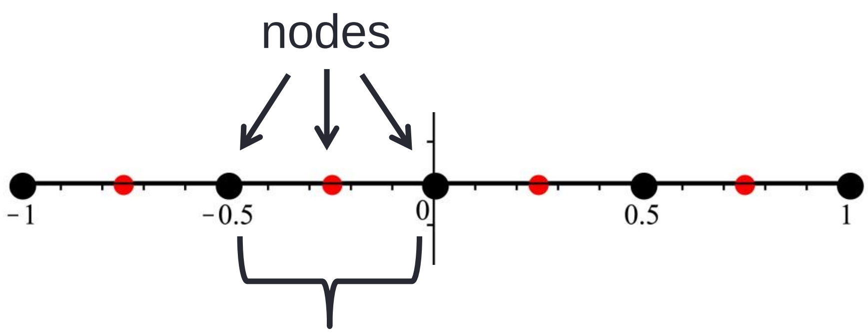
Considering Timoshenko's beam theory and linear elastic materials, the strain energy has the following form:

$$U = \frac{1}{2}EA\int_L \left(\frac{\partial u_0}{\partial x}\right)^2 dL + \frac{1}{2}EI_y\int_L \left(\frac{\partial \phi_y}{\partial x}\right)^2 dL + \frac{1}{2}\lambda GA\int_L \left(\left(\frac{\partial w_0}{\partial x}\right)^2 + 2\frac{\partial w_0}{\partial x}\phi_y + {\phi_y}^2\right)dL$$

A is cross sectional area,
I_y is second moment of area,
L is length of the beam.

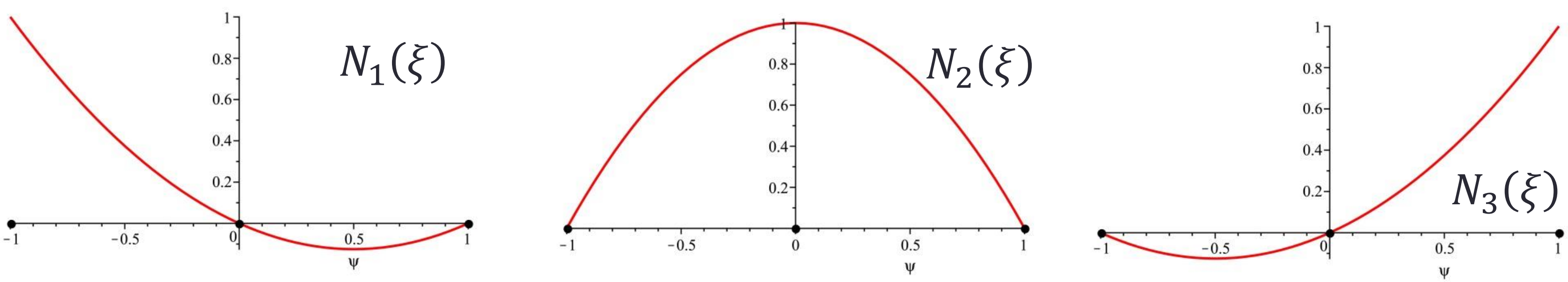
Timoshenko beam model

Finite element discretization of the beam.



The software Plaxis provides the following information at each node::

u_x: displacements in direction x
u_z: displacements in direction z
ϕ: rotation
N: Normal force
G: Shear force
M: Bending moment



Quadratic shape functions are used for each element

Timoshenko beam model

We approximate the functions $u_0,\ w_0,\ \phi_y$ by using quadratic finite elements. The finite element, given by nodes x_1, x_2, x_3 is transformed into the standard element $[-1;1]$ by the transformations

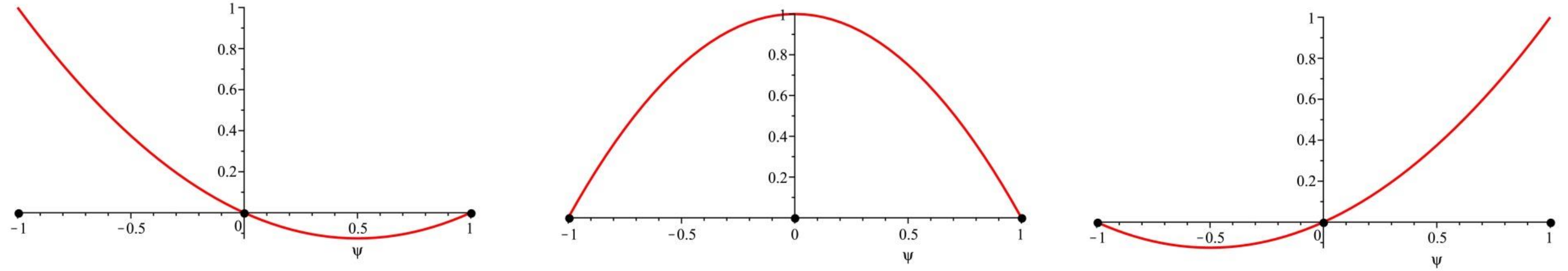
$$x = \frac{l}{2}\xi + \frac{(x_1+x_3)}{2}, \quad dx = \frac{l}{2}d\xi, \quad l = x_3 - x_1$$

$$\xi = \frac{2x}{l} - \frac{(x_1+x_3)}{l}, \quad d\xi = \frac{2}{l}dx$$

ξ is the local coordinate
x is the global coordinate

The shape functions N_1, N_2, N_3 for the standard element are:

$$N_1 = -\frac{1}{2}(1-\xi)\xi, \quad N_2 = (1+\xi)(1-\xi), \quad N_3 = \frac{1}{2}(1+\xi)\xi$$



Timoshenko beam model

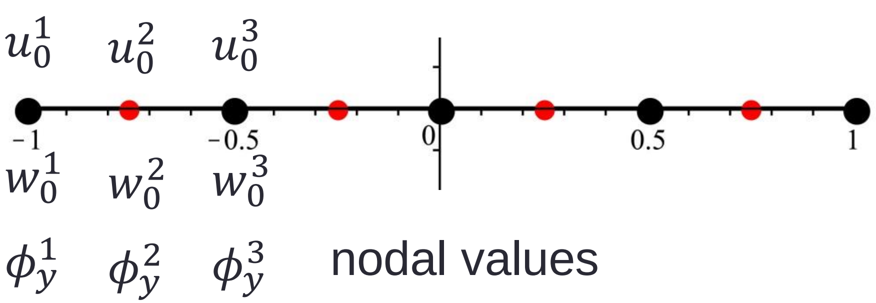
Let u_0^1, u_0^2, u_0^3 are the values of u_0 at the nodes of the finite element. Then, the displacements and the rotation can be expressed by the nodal values and the shape functions:

$$u_0(\xi) = u_0^1 N_1 + u_0^2 N_2 + u_0^3 N_3$$

$$\frac{\partial u_0}{\partial \xi} = u_0^1 \frac{\partial N_1}{\partial \xi} + u_0^2 \frac{\partial N_2}{\partial \xi} + u_0^3 \frac{\partial N_3}{\partial \xi}$$

$$w_0(\xi) = w_0^1 N_1 + w_0^2 N_2 + w_0^3 N_3$$

$$\phi_y(\xi) = \phi_y^1 N_1 + \phi_y^2 N_2 + \phi_y^3 N_3$$



Timoshenko beam model

For computation of the integrals for the energy *U* on every element we use the 3-point Gaussian quadrature formulas.

$$\int_{-1}^{1} f(\xi)d\xi = c_1 f(\xi_1) + c_2 f(\xi_2) + c_3 f(\xi_3)$$

with nodes $\xi_i = -\sqrt{0.6},\ 0, +\sqrt{0.6}$ and coefficients

c_i = 5/9, 8/9 and 5/9.

Timoshenko beam model

The strain energy, written in global coordinate system, has the following form:

$$U = \frac{1}{2} EA \int_L \left(\frac{\partial u_0}{\partial x} \right)^2 dL + \frac{1}{2} EI_y \int_L \left(\frac{\partial \phi_y}{\partial x} \right)^2 dL$$
$$+ \frac{1}{2} \lambda GA \int_L \left(\left(\frac{\partial w_0}{\partial x} \right)^2 + 2 \frac{\partial w_0}{\partial x} \phi_y + {\phi_y}^2 \right) dL$$

The strain energy is obtained by the sum of the strain energies of each element in the local coordinate system:

$$U = \frac{1}{2} \sum_e \left(EA \int_{-1}^{1} \left(\frac{\partial u_0}{\partial \xi} \right)^2 \frac{2}{l} d\xi + EI_y \int_{-1}^{1} \left(\frac{\partial \phi_y}{\partial \xi} \right)^2 \frac{2}{l} d\xi + \lambda GA \int_{-1}^{1} \left(\frac{\partial w_0}{\partial \xi} \right)^2 \frac{2}{l} d\xi \right.$$
$$\left. + 2\lambda GA \int_{-1}^{1} \frac{\partial w_0}{\partial \xi} \phi_y d\xi + \lambda GA \int_{-1}^{1} {\phi_y}^2 \frac{l}{2} d\xi \right)$$

Expected Results

- Show a way to calculate the **strain energy** of one beam element

- Show a way to calculate the **second order work** for this kind of beam element with the data given

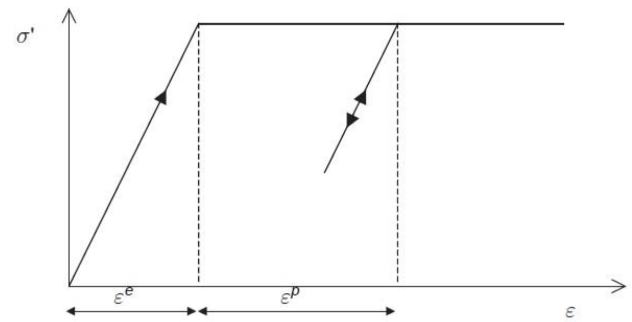
Second order work

The **second order work** can be obtained by

- computing the strains,
- checking if the beam is in elastic or plastic regime,
- obtaining the stresses and
- computing the second order work by the computed strains and stresses:

1. The strains in each node,
can be computed from:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_y}{\partial x}$$
$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w_0}{\partial x} + \phi_y$$



elastic-plastic stress-strain relations

2. By knowing the elastic-plastic stress-strain relations,
the stresses σ_x and τ_{xz} can be determined.

3. If the stresses and the strains are known,
the second order work is given by:

$$u = d^2W = d\sigma\colon d\varepsilon$$

Thank you for your attention!