Mathematical Model of Residential Storage Water-heating System

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Abstract

The present document reports the work carried on during ESGI 120 in Sofia. The considered problem is *Mathematical Model of Residential Storage Water-heating System*. We strove to model the device from Melisa Climate Ltd which controls the water heater system in the house. We used two approaches to model the problem that complement each other. The first one represents the dynamics inside the heater and the second model looks at the control loop run by the device without assuming any delay caused by first block. Although an integrated model is not developed, the ingredients for such are presented.

Key words: convection/diffusion heat transfer, Boussinesq approximation, feed-back control-loop

1 Introduction

The low-cost commonly available water heater is inherently a lossy device and this means that a high percentage of the energy consumed for heating water ends up being lost as heat to the surroundings. Informal measurements and practice show that when a family of four people switches on and off the water heater just when required, savings of up to 80% are recorded.

Our goal is to develop a model of the water heater to perform energy saving calculations and help the customer to operate their water heaters in the most energy-efficient way.

1.1 Definition of the problem (Melissa Climate)

An electric water heater consists of: an inner steel tank, that holds the water being heated, insulation that surrounds the tank so as to decrease the amount of heat loss, pipe to allow cold water to enter the tank, pipe to allow hot water to leave the tank, thermostat that reads and controls the temperature of the water inside the tank, heating element that heats the water by means of electricity and other components for safety and maintenance.

Water temperature inside the heater is controlled by the mechanical thermostat. The temperature may usually be set by the user somewhere in between 40 and 70°C. A microcontroller is used to gather real-time information from a water heater.

The information is collected by different sensors (Fig. 1) and consists of data about: current temperature of the cold water pipe, temperature of the hot water pipe, environment temperature (home temperature), electric current and voltage. Important notice the temperature sensors of the cold and hot water pipes are installed ONTO the pipe itself. There is significant temperature loss depending on the pipe diameter, material and others.

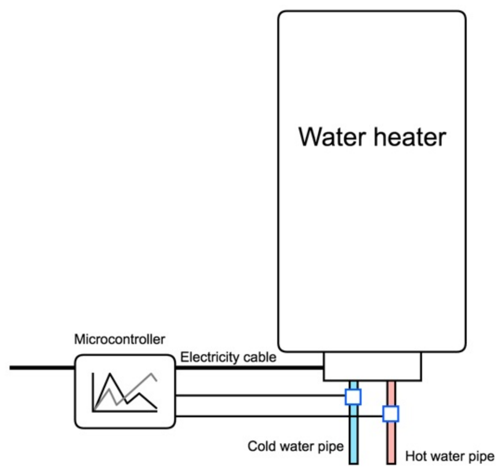

Figure 1: Schematic view of the heater system

1.2 Overview

The problem was observed from two different but complementary perspectives. First, a *closed-flow* problem was considered in order to understand the dynamics inside the heater (Section 2). The process is simplified to a 1-dimensional problem along the height of the water tank and spacial temperature distribution is investigated. Then, in Section 3, we analyzed the temperature dynamics during water consumption, assuming uniform temperature distribution inside the heater. The realistic solution would be a combination of both models, since the time required to reach uniform temperature in the second model should be estimated from the first one. Moreover, the assumption of uniform temperature inside the tank does not appear to be imposing any stringent premises and the results in Section 3 seem to comply with those provided by the company.

2 Temperature dynamics inside the Water Heater

In this section, we develop a model of the temperature distribution inside the water tank. Consider the case when the inflow and outflow valves are closed. The heater is turned on at time t_0. Changes in temperature impose a density variations along the tank, which drives a water flow due to natural convection. The density difference is proportional to the temperature difference as:

$$\rho - \rho_0 = -\beta(T - T_0), \tag{1}$$

where β is thermal expansion coefficient, T and ρ are fluid temperature and density, and T_0 and ρ_0 are reference values for them. The fluid motion is then governed by Navier–Stokes and continuity equations for incompressible flow, and Boussinesq approximation (see [5]) is used to take into account density differences. The temperature variation in the domain is described with a heat equation. Therefore, the governing equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\boldsymbol{\nabla} \cdot \mathbf{v} = -\frac{1}{\rho_0}\boldsymbol{\nabla} P + \nu \boldsymbol{\nabla}^2 \mathbf{v} - \mathbf{g}\beta(T - T_0), \tag{2a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{2b}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\boldsymbol{\nabla} T = \alpha \boldsymbol{\nabla}^2 T, \tag{2c}$$

where $\mathbf{v}$ is fluid velocity, P is pressure, ν is fluid viscosity and α is thermal diffusivity. The constants used in Eq. (2) and in other equations are given in Table 1.

Eq. (2) are too complex to attack directly. We assume that the domain is 1-dimensional with spacial dimension x along the vertical axis of the tank. $x = 0$ corresponds to the bottom of the tank and $x = L$ to the top part. We assume in Section 2.1 that the heat is supplied at $x = 0$. Therefore, the hot bottom plate is heating up the cold water.

To avoid solving Navier–Stokes equations in Eq. (2) for the fluid motion inside the water tank we use an estimation for the spatial velocity distribution of the fluid. Since, we are looking at a closed-flow system, the velocity on top and bottom of the heater is zero and we assume parabolic flow profile:

$$v(x) = \frac{4x(L - x)}{L^2} v_{\max}, \tag{3}$$

where we estimated $v_{\max} = \sqrt{2gL\beta\Delta T}$ by equalizing the pressure difference $\Delta p = \rho\beta\Delta T gL$ to the dynamic head $\rho v^2/2$ in absence of viscous forces in the flow.

Table 1: *Values of the physical parameters*

Constant	**Value**
ρ	988 kg / m^3
c_p	4185 J /(kg .° C)
λ	0.59 W /(m.° C)
V	225 m^3
I	{0; 13.5}
κ	1.4269 e-07 W. m^2 / J
L	1 m
$T_{\mathbf{f}}$	500 s
V_0	0.1963 m^3
c_m	{0; 0.0211}
T_{A}	25 ° C
T_{H}	70 ° C

Note that this velocity choice does not satisfy the continuity equation in Eq. (2). This is a drawback of our model, however, we reason that this velocity is more realistic than simply a constant one (which satisfies the continuity equation).

2.1 Simplified model

After imposing the velocity distribution v for the fluid motion, we are left only with the heat equation (2c) from the original system Eq. (2). To close the problem, we need to impose initial and boundary conditions. The boundary conditions for this case would be Dirichlet for the lower plate

$$T\Big|_{x=0} = T_{\mathrm{H}}. \tag{4}$$

and Neumann (no heat loss) for the upper plate

$$\frac{\partial T}{\partial x}\Bigg|_{x=L} = 0, \tag{5}$$

where $T_{\mathrm{H}} = 70^{\circ}$ C is the temperature of the hot water at the bottom. The initial condition is chosen to agree with boundary conditions at $t = 0$ and has a shape of piecewise linear function:

$$T_0(x) = \begin{cases} \dfrac{(T_{\rm H} - T_{\rm A})(x - 0.1)}{0.1} + T_{\rm A}, & 0 \le x < 0.1, \\ T_{\rm A}, & x \ge 0.1, \end{cases} \tag{6a}$$

where $T_{\rm A} = 25°$ C is the ambient temperature, i.e. we are taking into account the heat loss from the heater.

We used finite difference approximation for the derivatives and thus to solve Eq. (6). A fully implicit scheme with 100 grid points in space and 5000 grid points in time was implemented.

Fig. 2 illustrates the temperature variation for a few points inside the tank and Fig. 3 shows all the temperature waves at all times. Depending on the location of the heater we refer as the temperature of the heater the amount of time we need for the water to heat up defers. In addition, the average speed of the temperature wave is much smaller than the maximum velocity. This suggests that for an inflow of water to heat up, there is a delay time one has to take into consideration. The delay temperature depends on the temperature of the inflow water.

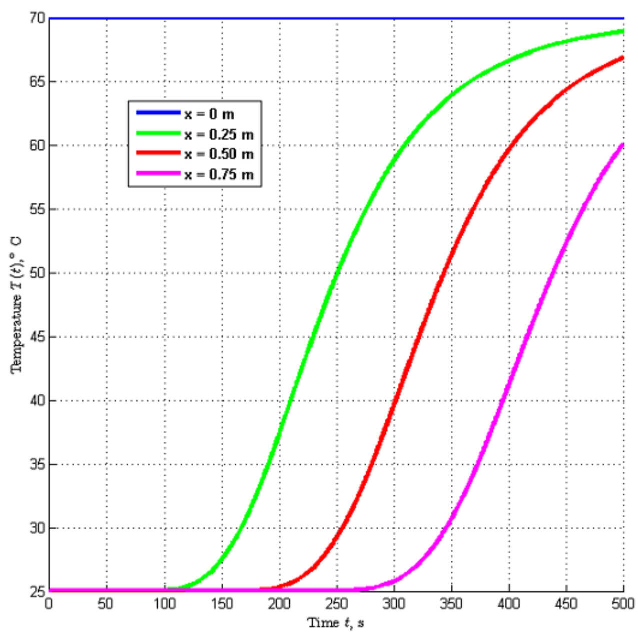

Figure 2: Temperature distribution for $v_{\rm max} = 0.003$ [m/s]

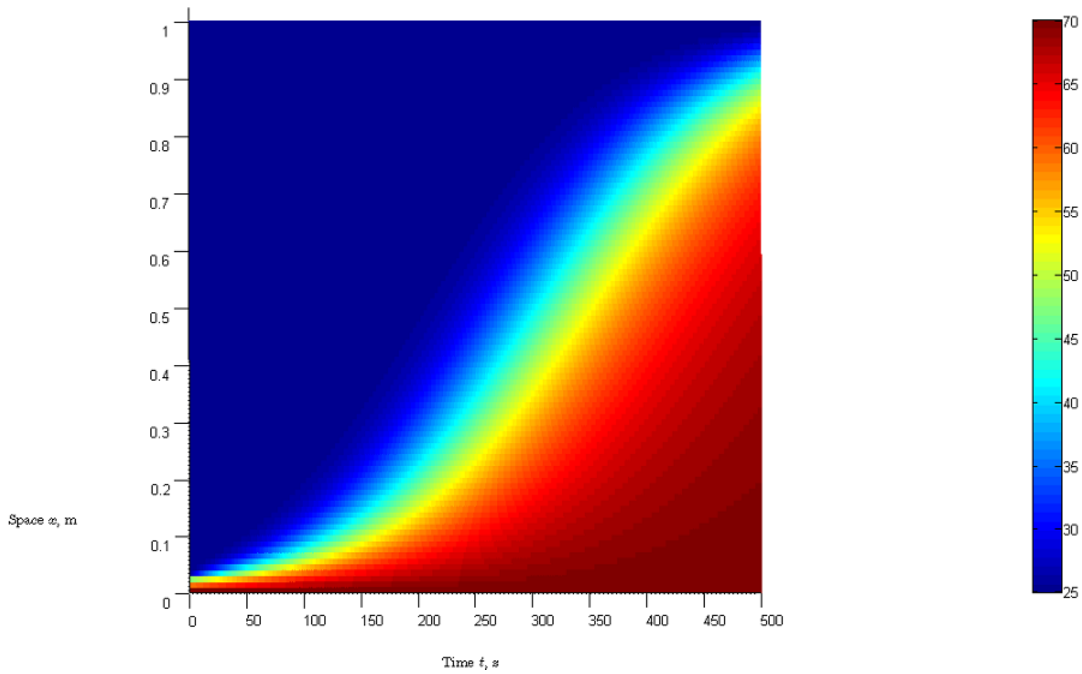

Figure 3: Temperature profile evolution of the heater

3 Water Heater Control System

Here we assume the water heater is uniform at all times. We also suppose there is no delay in water heater system. The more realistic model shall be constructed by relaxing the later assumption. The general control block of the control system looks in figure 4.

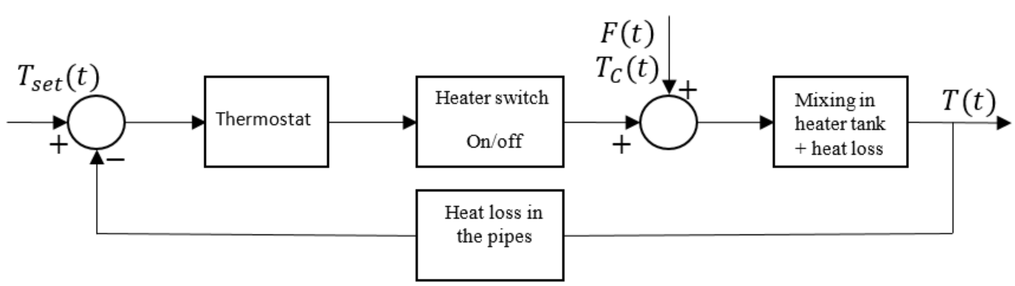

Figure 4: General control loop scheme for water heater system

Here we don't take cold water variations into account therefore $T_c(t) \equiv T_c$. Also we consider the case which the desired temperature is set to a constant value $T_{set}(t) \equiv T_h$. The dynamics of the thermostat and the heater power, $P(t)$, can be described as a function of temperature and time:

$$P(t, T_h) = \begin{cases} VI, & T_{set} - T_h > \epsilon, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

i.e. we assume either the heater works with full power or it is off.

For the flow of cold water we assume that it is a piecewise constant function, that is, the tap is either full open or full closed and also the consumption (flow) is the same whenever the tap is open. Therefore, the flow function could be described as

$$F(t) = \sum_i f_i \chi(t; t_i, t_{i+1}), \tag{8}$$

where

$$\chi(t; t_i, t_{i+1}) = \begin{cases} 1 & t \in [t_i, t_{i+1}] \\ 0 & otherwise \end{cases}, \tag{9}$$

and $f_i = r_i F$ with $r_i = \{1, 0\}$ and $F = 2\, liter/min$.

Thus the overall heat balance around the system reads (similarly to [1]-[4])

$$\frac{dE(t)}{dt} = \frac{Mc_p d(T_h(t) - T_c)}{dt} = P(t) + F(t)c_p(T_h(t) - T_c) + \frac{(T_h(t) - T_a)A}{R}, \tag{10}$$

with $E(t)$ being the internal energy, T_a room temperature, t_d time delay due to the heating of the water and R thermal resistance of the tank.

To approach this problem, we chunk the time domain into finite small intervals. The quality of the solution clearly depends on resolution of the mesh. Due to Eqs. (7) to (9), since at each time interval the coefficients in Eq. (10) are constant, Eq. (10) turns into

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_i y(t) = b_i, \quad t_i \leq t < t_{i+1} \tag{11}$$

which has analytical solution in the form

$$y(t) = \xi_i \mathrm{e}^{-a_i t} + \frac{b_i}{a_i}, \quad t_i \leq t < t_{i+1}, \tag{12}$$

where ξ_i depends on the solution at t_i from the previous time step (i.e, $t_i \leq t < t_{i+1}$).

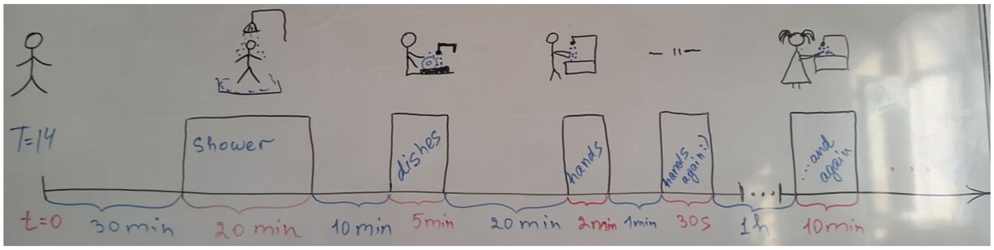

Figure 5: A sample usage of water in a day

As an example we assumed a consumption schedule for a person in a day time as in Fig. 5. We took the sensitivity of the thermostat to be 3°C. Fig. 6 illustrates the history of the electric power and the temperature profile for the outlet flow from $t = 30$min onwards.

Finally, we need to take into account for the heat loss from the pipes that occurs between the heater and the sensors. According to problem description, the temperature that is measured by the sensor is on the pipe. Therefore, there is a convection heat transfer from the fluid inside the pipe to the inner wall of the pipe which is followed by a conduction heat transfer from there to the outside surface of the pipe. Finally, the outer surface is cooled down by Newton cooling law through the cold surrounding air. The heat balance for the two processes are expressed as

$$h_{water}A_{inner}(T_{water} - T_1) = \frac{k}{d}(T_1 - T_2), \tag{13}$$

$$h_{air}A_{outer}(T_2 - T_{air}) = \frac{k}{d}(T_1 - T_2), \tag{14}$$

where d is the thickness of the tube and T_1 and T_2 are the temperatures of the inner and outer surface of the pipe, respectively. Using the proper parameters for h_{water} and h_{air} and properties of typical pipes used in houses (e.g. thickness=2mm, diameter=1inch) we found a decrease of maximum 3°C. An improved model would be to plug Eqs. (13) and (14) into Eq. (10) so that the heat loss plays its role in the dynamics of the tank.

4 Conclusions

During the week of ESGI 120 we developed and solved the model for temperature dynamics the residential electrical water heater. Two proposed approaches were

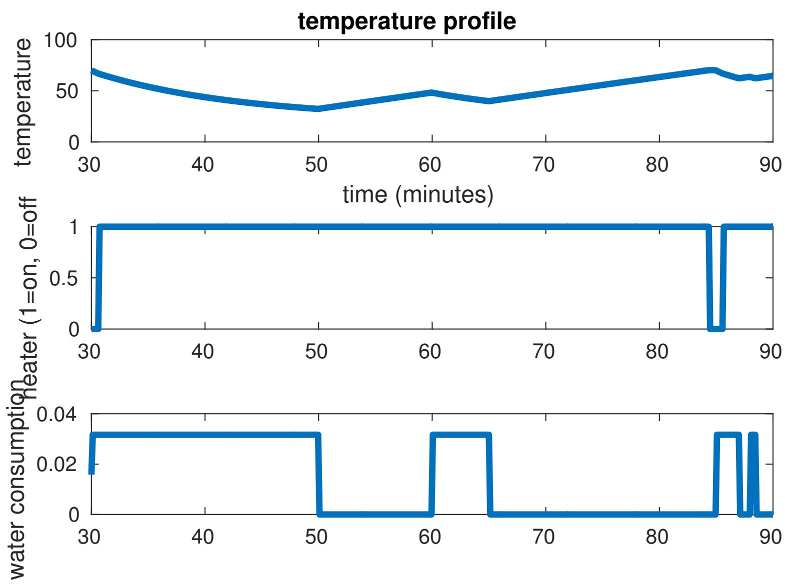

Figure 6: Hot water usage, power and temperature history for toy problem from Fig. 5

covered. First, we constructed a 1-D spacial model of the temperature distribution inside the water tank with the heater located at the bottom of it. The 1-D approximation allowed us to avoid heavy numerical simulations, however, it has couple of limitations such as the location of the heater is strictly at the bottom of the tank and imposed fluid velocity that does not satisfy the continuity equation. The later limitation can be withdrawn if one constructs 2-D model. The results and parameter estimation suggest domination of the convective effect with respect to diffusion, and therefore, suitable choice of the velocity profile is crucial for the model.

The second proposed model described the temperature dynamics in the tank. This model fits better to describe the experimental data, however, also has its limitations, such as an assumption of uniform temperature inside the tank which results into neglecting the time which is required for the water on the top of the tank to heat up.

Both models can be integrated into a single complete model, and it is the proposed step for the further research.

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