Post-Processing for Beam Elements: Calculating the Second Order Work and Strain Energy

1 Company's Overview

gruner|> The Basel-based Gruner Group is an independent market leader in engineering and planning consultancy services. Founded over 150 years ago, we now have more than 20 companies at over 30 sites in Switzerland and all over the world.
Gruner is a byword for construction services that set benchmarks in terms of quality. A broad set of competencies, a committed workforce, in-depth expertise and many years of project experience enable us to provide support for complex building projects throughout their entire life cycle. The interdisciplinary teamwork adopted by Group companies guarantees that customers receive individual solutions for challenging construction projects.
Through our cooperation with both higher education institutions and public and private research teams, our knowledge is in a constant state of development.

2 Definition of the problem

Modern design of excavation pits includes often the simulation of the problem using Finite Element (FE) models. Primary objective for this kind of models is to predict deformations of the excavation pit support and the surroundings. As a secondary objective, it would be very useful to know, if the modelled structure is near to a failure state or not.

For this kind of modelling task, usually a commercial software FE package is used as a black-box model with no access to the source code and no option for altering the algorithms included in the code. This is the case because this kind of software packages are widely known, well tested and accepted by the professional engineer community.

To model excavation pits and especially their support usually a plain-strain analysis is performed where beam elements are employed. The software package used by the company (Plaxis [2]), and this FE code is widely used in geotechnical engineering. This software package provides a beam element based on Mindlin's theory of plates.

3 Task description

A well-known criterion for failure (bifurcation in the solution) for models using elasto-plastic formulation is the Hill's failure [1] criterion based on the second order work. The second order work is defined at the material point by:

$$u = d^2W = d\sigma : d\varepsilon$$

$$[u] = \frac{N}{m^2} = \frac{J}{m^3}$$

Hill's condition of stability (Hill 1957) is defined by:

$$d^2W > 0$$

A related measure is the strain energy, which is the integrated value of the second order work over a volume V:

$$U = \int_V u \, \mathbf{d}V = \int_V d\sigma : d\varepsilon \, \mathbf{d}V$$

$$[U] = J = N \cdot m$$

Unfortunately, the software package Plaxis does not calculate the second order work. Furthermore, it only provides a small set of output variables at the element nodes. The objective is to calculate the second order work for this kind of beam elements as a post-processing after finishing the FE-simulation based only on the nodal variables provided in the programs output. The output provided at each node:

- u_x: total nodal displacements in direction x
- u_y: total nodal displacements in direction y
- φ: total nodal rotation
- N: Normal force extrapolated to the node*
- G: Shear force extrapolated to the node*
- M: Bending moment extrapolated to the node*

* The extrapolation function is unknown.

From NFEM Ch11 [3] we found a formula to calculate the strain energy W for a Timoshenko beam element. Is it also valid for a beam based on Mindlin's theory of plates? If not, please provide the correct solution.

$$W = \int_{L0} \left(N^{\mathbf{0}} \cdot e + \frac{\mathbf{1}}{\mathbf{2}} E \cdot A_{\mathbf{0}} \cdot e^2\right) d\bar{X} + \int_{L0} \left(G^{\mathbf{0}} \cdot \gamma + \frac{\mathbf{1}}{\mathbf{2}} G \cdot A_{\mathbf{0}} \cdot \gamma^2\right) d\bar{X} + \int_{L0} \left(M^{\mathbf{0}} \cdot \kappa + \frac{\mathbf{1}}{\mathbf{2}} E \cdot I_{\mathbf{0}} \cdot \kappa^2\right) d\bar{X}$$

4 Expected Results

- Show a way to calculate the second order work for this kind of beam element with the data given
- Show a way to calculate the strain energy of one beam element

Literature:

1. R. Hill, On uniqueness and stability in the theory of finite elastic strain. Journal of the Mechanics and Physics of Solids", V.5, 4, pp. 229 – 241, 1957.

   http://www.sciencedirect.com/science/article/pii/0022509657900169

2. Information from the Plaxis manual on beam element definition based on Mindlin plate theory.
3. http://www.colorado.edu/engineering/cas/courses.d/NFEM.d/NFEM.Ch11.d/NFEM.Ch11.Slides.d/NFEM.Ch11.Slides.pdf